# LinoSwap: A Rarity-Friendly NFT Automated Market Maker

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**Abstract**

NFT Automated Market Maker is the ultimate bridge between NFT and DeFi. It will provide unlimited possibilities for NFT-Fi and accelerate the NFT revolution rapidly. In this paper, we present a new NFT AMM protocol that is suitable for NFT assets with different rarity. NFTs of arbitrary rarity will have fair prices when traded using our protocol. The protocol consists of three parts: the Basic Curve, the Rarity Function, and the Rarity-Value Curve. We derive our Basic Curve from the constant product model, the Rarity Function from existing rarity tools, and the Rarity-Value Curve from the Pareto distribution. Moreover, through theoretical analysis, we study the key parameters, e.g., the max relative value $V_{max}$ and clarify their impact on our protocol. Our analysis is not only indispensable to our protocol but also helpful to future NFT AMM and NFT-Fi studies.

## 1 Introduction

NFT AMM is the most important paradigm for NFT trading in the future, which can bring the following benefits:

- Complete decentralization of NFT trading: truly NFT DEX.
- Qualitative improvement of NFT liquidity: traders can immediately sell NFTs.
- The ultimate bridge between NFT and DeFi: it will provide unlimited possibilities to NFT-Fi.

But the existing NFT AMMs have one biggest drawback which is ignoring differences between items in the same NFT collection. The difference between NFT items mainly comes from their traits. Considering simplification, the difference in traits can be compressed into the difference in rarity.

If the swap output of a high-rarity NFT and a low-rarity NFT is the same, and if the user is economically rational, then eventually only NFTs near the floor price value will be used to swap. For the high-rarity NFTs, there will be a negative difference between the actual value and the value recognized by NFT AMM. Moreover, if the value recognized by NFT AMM is not much bigger than that of the orderbook model's best offer, the advantages of NFT AMM cannot be reflected—users can also choose to accept an offer to sell NFTs immediately.

We implement a rarity-friendly NFT automated market-making protocol by introducing the Rarity Function and Rarity-Value Curve on top of AMM's Basic Curve so that high-rarity NFTs in the same collection can be accepted by AMM at a fairer price. At the item level, our protocol can fit a larger range of rarity within the same collection. At the collection level, our protocol can also fit most NFT collections. And our protocol is also simple enough for traders to understand.

The rest of this paper is structured as follows:

In section 2 we introduce and analyze some properties of the Basic Curve, which is very close to the Bonding Curve or Conservation Function used by DeFi AMMs.

In section 3 we introduce and analyze the Rarity Function. We decided to use the Rarity Score Rank Percentage as our rarity quantification method, and first use data from some third-party rarity tools as input to our Rarity Function.

In section 4 we introduce and analyze the Rarity-Value Curve. We decide to use the Pareto distribution as the Rarity-Value Curve first and analyze some trade-offs in choosing the parameter $V_{max}$ for the Rarity-Value Curve based on the High Rarity Liquidity Impact Factor and AMM Rarity Tolerance.

# 2 Basic curve

Suppose there is a pool consisting of two assets, token X and token Y, where token X is NFT(ERC721) with reserve $x$, and token Y is FT(ERC20) with reserve $y$. Our basic curve is based on the bonding curve of the constant product model.

## 2.1 Conservation function

For the constant product model, the conservation function(or bonding curve) is:

$$x \cdot y = L \tag{1}$$

Where $L$ is defined as the amount of liquidity:

$$L = x_0 y_0 \tag{2}$$

Where $x_0$ and $y_0$ are the initial reserve of token X and token Y in the pool, respectively. It's worth noting that the token X reserve $x$ is somewhat continuous.

When an NFT is added to the pool, the change it makes to the reserve of token X is $\Delta x$: $\Delta x = relative\ value\ of\ the\ item\ \times\ benchmark\ reserve$. Where relative value is determined by rarity, and benchmark reserve is a constant that equals 1(we'll discuss more about this in Section 3 and Section 4).

## 2.2 Price Curve

For the constant product model, the price-reserve curve is:

$$P(x) = -\frac{dy}{dx} = \frac{L}{x^2} \tag{3}$$

In the initial case, $x = x_0$, therefore:

$$P(x_0) = \frac{L}{x_0^2} = \frac{y_0}{x_0} \tag{4}$$

# 3 Rarity Function

## 3.1 Analysis of Rarity

The property of non-fungible of NFTs essentially comes from the differences in traits. The degree of non-fungible differences between NFTs is a function of traits. When traders trade NFTs, differences in their estimates of the value of different items in the same NFT collection lead to different sale prices for NFT items. Non-fungible differences can be identified and utilized by AMM protocols—estimate of value by the algorithm.

People tend to keep things simple, therefore the degree of non-fungible differences can be simplified to a univariate function—rarity. And the relationship between rarity and sale price is generally monotonic.

From rarity to value, we need to solve the following 2 problems:

1. The same collection, given NFT Item, obtain the rarity.
2. The same collection, given the rarity, calculates the relative value.

The relative value refers to the value ratio of the NFT to the benchmark NFT. While the benchmark NFT refers to the NFT with the lowest rarity in the collection.

In this section, we will analyze and solve the first problem, and in the next section, we will analyze and solve the second problem. We hope our solutions to these 2 problems generalize to the vast majority of NFT collections.

## 3.2 Rarity Score Rank Percentage

There are many methods to calculate the rarity of NFTs from the degree of non-fungible differences. And the rarity of distinct items is not the same if the methods are different. Using a score to describe rarity is a common method, the score $r$ of Rarity.Tools is calculated as:

$$r = \sum_{i=1}^{n} \frac{1}{t_i} \tag{5}$$

Where $t_i$ represents the ratio of the number of items with the same value of the $i$-th trait to the total number of items. For example, if the value of a trait $m$ is owned by a total of 1000 items, and the total number of items is 10000, then $t_m = 0.1$ and the rarity score of this trait is 10. But this was the calculation method used by Rarity.Tools.

Rarity calculation methods of many rarity tools are opaque. Rarity tools might work closely with NFT collection creators to determine the rarity score for some items. To make our protocol more adaptable, we conduct the following analysis.

Let the rarity score distribution of an NFT collection be(for simplicity, we consider it as a continuous function):

$$F(r) = \int_{-\infty}^{r} f(t)dt \tag{6}$$

Where $F(r)$ is the probability distribution function of rarity $r$, $f(t)$ is the corresponding probability density function and satisfies $\int_{-\infty}^{+\infty} f(t)dt = 1$.

We can find that if we use the rarity score to denote $r$, then the $r$ of items in different NFT collections is not comparable, therefore we have to normalize every rarity score function to make it work with our basic curve. In order to make it more adaptable, considering simplification, we replace the rarity score with the rarity score rank percentage. For rarity score rank percentage data $r_i$, it can be obtained by:

$$r_i = R(i) \tag{7}$$

Where $R$ is the Rarity Score Rank Percentage Function(or Rarity Function for short), and $i$ is the token ID of the NFT item.

The Rarity Function can be implemented based on data from third-party rarity tools. However, we should pay attention to rarity data from third-party rarity tools, whose generation might be opaque and storage might be centralized. We will realize on-chain rarity and decentralized rarity solutions in the future to solve this problem thoroughly.

# 4 Rarity Value Curve

## 4.1 Analysis of Rarity Value

After we have identified the rarity function that can easily adapt to most NFT collections, we are one last curve away from the completeness of our protocol: the Rarity-Value Curve.

Rarity-Value Function describes the relationship between rarity score rank percentage and relative value. The relative value $p_r$ can be obtained by Rarity-Value Function $V$:

$$p_r = V(r) \tag{8}$$

By combining the rarity-value function and rarity function, in the same NFT collection, we can obtain the relative value $p_r$ of an item compared to the benchmark item:

$$p_r = V(r_i) = V\big(R(i)\big) \tag{9}$$

Moreover, combined with the basic curve, in the same collection, the price $p$ of arbitrary NFT items after considering rarity is:

$$p = p_r \cdot P(x) = V\big(R(i)\big) \cdot P(x) \tag{10}$$

Where $i$ is the token ID of an item, and $P(x)$ is the price function in the basic curve, which determines the benchmark price(or basic price) for the collection. Naturally, the benchmark price equals the price of the benchmark item after considering rarity. In addition, for an NFT item $i$ with rarity $R(i)$, $p_r$ also denotes how many benchmark items it's equivalent to.

## 4.2 Pareto Distribution

The first rarity-value curve we will support is derived from the Pareto distribution, which we believe may be the closest to the natural rarity-value distribution in most cases(and is not very complex).

Pareto distribution is named after Vilfredo Pareto—the Italian civil engineer, economist, and sociologist, which is a power-law probability distribution that is used in the description of social, quality control, scientific, geophysical, actuarial, and many other types of observable phenomena. The famous "80-20 rule" is derived from the Pareto distribution.

The probability distribution function of Pareto distribution is:

$$F(r) = \left(\frac{r}{r_0}\right)^{-k} \tag{11}$$

And the corresponding probability density function is:

$$f(r) = \frac{kr_0^k}{r^{k+1}} \tag{12}$$

Where $r \in [r_0, 1]$, is the rarity score rank percentage. For an NFT collection with $N$ items, $r_0 = 1/N$. In order to eliminate the influence of $N$ on the rarity-value curve, considering simplification, we fix $N$ to 10000, then $r_0$ is fixed to 1/10000 (0.01% as base point), which also means that the minimum step size of $r$(rarity's accuracy) is 0.01%.

Let $R = 10000r,$ we have:

$$f(R) = \frac{kR_0^k}{R^{k+1}} \tag{13}$$

Where $R_0 = 1$(corresponding to $r_0$=0.01%).

When $R$ takes the maximum value $R_{max} = 10000$(corresponding to $r_0$=100.00%), we have:

$$f(R)_{min} = \frac{k}{10000^{k+1}} \tag{14}$$

We want to let $f(R)_{min} = 1$ because the value of the rarity-value curve is relative to the value of the benchmark NFT item, but find that $k$ is unsolvable. Consider modifying our protocol, the simplest change is to multiply it by a constant $k'$ such as that:

$$k' \frac{k}{10000^{k+1}} = 1 \tag{15}$$

The new function is:

$$V(R) = f'(R) = \frac{k'kR_0^k}{R^{k+1}} = \left(\frac{10000}{R}\right)^{k+1} \tag{16}$$

$V(R)$ is our Rarity-Value Function or Rarity-Value Curve. When $R$ takes the minimum value $R_0 = 1$, we have:

$$V_{max} = 10000^{k+1} \tag{17}$$

## 4.3 High Rarity Liquidity Impact Factor

$V_{max}$ is a crucial parameter that determines the ratio of the highest-valued NFT to the lowest-valued(benchmark) NFT in the same collection. Because our AMM protocol will automatically complete the conversion of high-rarity NFTs into FTs, we must consider the impact of high-rarity NFTs on the liquidity of the AMM pool.

Selling a single high-rarity NFT may consume most of the FTs in the pool, causing the pool to tilt sharply. We call this phenomenon High Rarity Liquidity Impact, which will greatly affect the follow-up trader's trading experience.

For a pool with token pair status of $(x, y)$, the maximum output of a swap with a single NFT is:

$$\Delta y_{max} = y - \frac{L}{x + V_{max}} = \frac{V_{max}}{x + V_{max}} y \tag{18}$$

$$\frac{\Delta y_{max}}{y} = \frac{1}{\frac{1}{\frac{V_{max}}{x}} + 1} \tag{19}$$

Let:

$$F = \frac{V_{max}}{x} \tag{20}$$

We call $F$ the High Rarity Liquidity Impact Factor. The larger the $F$, the greater the probability of high rarity liquidity impact.

## 4.4 Rarity Tolerance

The smaller the $V_{max}$ the smaller the probability of high rarity liquidity impact, but this will make sellers with high rarity NFTs dissatisfied, and eventually give up using AMM for trading.

We define the highest rarity that a seller is willing to trade with AMM as the AMM Critical Rarity. Then we define the interval from minimum rarity to AMM critical rarity as AMM Rarity Tolerance Interval. Finally, we define the length of AMM Rarity Tolerance Interval as AMM Rarity Tolerance $T$:

$$T = \frac{R_{min} - R_c}{R_{min} - R_0} \times 100\% = \frac{1 - R_c}{1 - \frac{1}{10000}} \times 100\% \tag{21}$$

Where $R_c$ is the AMM critical rarity, in the form of rarity score rank percentage.

From the perspective of seller psychology, the larger the rarity tolerance the wider the range of NFT rarity that AMM can be applied to. Our ultimate goal is $T \to 100\%$.

## 4.5 Vmax Trade-Off

The larger the $V_{max}$, the larger the high rarity liquidity impact factor, the greater the probability of high rarity liquidity impact, the larger the AMM rarity tolerance, and the wider the range of NFT rarity that AMM can be applied to, and vice versa.

We have to make a trade-off between reducing the high rarity liquidity impact factor and improving AMM rarity tolerance. And we will allow the pool creator to change the $V_{max}$ in the future. The setting of the $V_{max}$ value needs to be driven by the market experience.

For some common $V_{max}$ values, the values of $k$ are shown in Table 1.

Table 1: Common $V_{max} - k$ examples

| $V_{max}$ | 1.5 | 2 | 3 | 4 | 5 | 10 | 100 | 1000 |
|---|---|---|---|---|---|---|---|---|
| $k$ | -0.9560 | -0.9247 | -0.8807 | -0.8495 | -0.8253 | -0.75 | -0.5 | -0.25 |

For $V_{max} \in \{2, 5, 10, 100\}$, the rarity-value functions are: $v = \left(\frac{10000}{R}\right)^{0.0753}$, $v = \left(\frac{10000}{R}\right)^{0.1747}$, $v = \left(\frac{10000}{R}\right)^{0.25}$, $v = \left(\frac{10000}{R}\right)^{0.5}$.

Some $R$ and $v$ of these four functions are shown in Table 2.

Table 2: $R$ and $v$ values of the four rarity-value functions

| $V_{max}$ \ $R$ | 1 | 2 | 5 | 10 | 20 | 50 | 100 | 200 | 500 | 1000 | 2000 | 5000 | 10000 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 2 | 1.89 | 1.77 | 1.68 | 1.60 | 1.49 | 1.41 | 1.34 | 1.25 | 1.19 | 1.13 | 1.05 | 1 |
| 5 | 5 | 4.43 | 3.78 | 3.35 | 2.96 | 2.52 | 2.24 | 1.98 | 1.69 | 1.50 | 1.32 | 1.13 | 1 |
| 10 | 10 | 8.41 | 6.68 | 5.62 | 4.73 | 3.76 | 3.16 | 2.66 | 2.11 | 1.78 | 1.50 | 1.19 | 1 |
| 100 | 100 | 70.7 | 44.72 | 31.63 | 22.36 | 14.14 | 10 | 7.07 | 4.47 | 3.16 | 2.24 | 1.41 | 1 |

We think that the most suitable $V_{max}$ value is not static, but changes with the market situation. If trading NFTs with AMM is very popular in the future, the $V_{max}$ can be 100 or even

1000, because high rarity liquidity impact is not that easy to happen. Moreover, when the liquidity pool is created, the higher the initial liquidity added to the pool, the larger the $V_{max}$ that could be set.

# 5 Miscellanea and Other Concerns

## 5.1 LP Token

LPs(Liquidity providers) that provide liquidity to a certain pool will receive a certain amount of LP tokens as proof of liquidity. Our LP tokens conform to the ERC20 protocol, so LPs can use them just like other ERC20 tokens—perform trading on Uniswap or lending on Aave.

When liquidity of $(x_0, y_0)$ is provided to the pool for the first time, the number of LP tokens obtained by LP is:

$$s_0 = y_0 \tag{22}$$

If it's not the first time to provide liquidity of $(\Delta x, \Delta y)$ to a pool(when token pair status is $(x, y)$, the total supply of LP tokens is $S$), then the number of LP tokens received by LP is:

$$s = \frac{\Delta y}{y} \cdot S \tag{23}$$

## 5.2 Fee

The initial version of our protocol supports a fixed fee of 2% for each pool but will support multiple fee rates in the future.

We support that pool LPs can claim fees without withdrawing liquidity. And the fee includes token pairs with the reserve $x_f$ and $y_f$, where the $x_f$ is the equivalent reserve of benchmark NFT. If $x_f$ is less than 1, it cannot be converted into a complete NFT. Therefore, we support swapping the fee of NFT into ETH when claiming.

## 5.3 Implementation

We will detail the implementation of the protocol(especially some core formulas) in future technical documentation.